UCSD/PTH 96–01

hep-ph/96mmnnn

SU(3) Decomposition of Two-Body B Decay Amplitudes

Benjamín Grinstein and Richard F. Lebed

Department of Physics, University of California at San Diego, La Jolla, CA 92093

(February 1, 1996)

Abstract

We present the complete flavor SU(3) decomposition of decay amplitudes for decays of the triplet (B_u^+, B_d^0, B_s^0) of B mesons nonleptonically into two pseudoscalar mesons. This analysis holds for arbitrarily broken SU(3) and can be used to generate amplitude relations when physical arguments permit one to neglect or relate any of the reduced amplitudes.

Typeset using REVTEX

I. INTRODUCTION

The current understanding of charge-changing quark transitions in terms of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix [1] is becoming progressively more open to scrutiny. Large numbers of new experimental results involving the physics of b-quarks permit one to perform incisive tests on the ill-known parameters of the third quark generation. The decays of B mesons should provide an ample testing ground for determining quantities of interest. Whether the CKM matrix is really unitary, the significance of the size hierarchy observed in CKM elements, and the origin of CP violation are issues that might be resolved, at least in part, within the next decade, owing to improved experimental information.

Before this optimistic program can be undertaken, however, one requires tools of analysis that facilitate the extraction of the relevant parameters from the data. The chief problem is that virtually all decays of B mesons contain at least one hadron, and fundamental quark-related quantities are notoriously difficult to extract from the corresponding hadronic quantities. Furthermore, the neutral B mesons B_d^0 and B_s^0 mix with their antiparticles, thus complicating particle identification. Nevertheless, one can make progress in both of these areas from knowing the symmetry of the underlying theory and a few of its dynamical properties.

A case in point is an interesting series of papers [2], in which it is claimed that one can disentangle CKM elements and strong-interaction final-state phase shifts from nonleptonic two-body B decays. The key ingredient in this analysis is the idea [3] that the numerous possible experimental measurements of nonleptonic decays of the mesons B_u^+, B_d^0, and B_s^0 can be related using the flavor SU(3) approximate symmetry of the strong interaction Lagrangian. Such a symmetry exists owing to the relative smallness of the u, d, and s quark masses compared to the QCD scale $\Lambda_{\rm QCD}$. According to Refs. [2], with *i*) a large number of these rates eventually measured, and *ii*) mild dynamical assumptions on the strong interaction physics interpreted at the quark level, one obtains through group theory an over-determined system of equations, which can be solved to isolate CKM elements and strong-interaction final-state

phase shifts. The B decays in this approach are described in terms of naive quark diagrams, some of which are taken to be suppressed on physical grounds; for instance, the diagram describing the annihilation of the valence quark-antiquark pair is said to be suppressed by a factor of the meson decay constant over its mass. Then, since the SU(3) flavor structure of the intermediate quark lines is simple, one can calculate a set of decay amplitudes in terms of reduced SU(3) amplitudes and look for relations between them. Because the coefficients are proportional to CKM elements and strong interaction phases, solving the system of equations permits one to extract these quantities. However, the quark diagram approach has two major drawbacks. First, the exact nature of SU(3) group theory is not fully manifest in such a description, so that relations thus derived tend to appear as surprising cancellations between diagrams; and second, the dynamical assumptions mentioned above are at best only semi-quantitative and do not lend themselves well to systematic corrections.

In this paper, we remedy the first problem by presenting the complete SU(3) decomposition for two-body nonleptonic decays of the B triplet to pseudoscalar mesons with and without charm, including arbitrary breaking of SU(3) (as well as isospin) symmetry. Many of the relations derived in Refs. [2] correspond to the suppression of Hamiltonian operators transforming under particular irreducible representations of SU(3), while others correspond to chance cancellations owing to the phenomenological neglect of particular quark diagrams in their description. The second problem will be addressed in a future publication [4]. The current paper is a reference work that permits one to perform the SU(3) analysis of these decays using any particular set of dynamical assumptions. Its chief advantage is that, while one can write down an Hamiltonian with an arbitrary number of parameters to produce a model, the number of SU(3) reduced amplitudes for given initial and final states is a finite and exactly calculable number, and all relations obtained due to symmetry alone are made explicit.

This paper is organized as follows: In Sec. 2, we describe two equivalent means by which one may obtain the necessary Clebsch-Gordan coefficients to decompose the physical amplitudes in terms of SU(3) amplitudes. In Sec. 3, we explain the counting of these

two types of amplitudes in fully broken SU(3). In Sec. 4 the means by which relations between physical amplitudes are obtained is explored. We consider examples arising from the assumption of an unbroken SU(3) Hamiltonian defined through a four-quark operator, as well as the inclusion of linear SU(3) breaking. Sec. 5 discusses directions for future work and concludes. The group-theoretical results are contained in the Appendix.

II. SU(3) GROUP THEORY

A full treatment of the SU(3) decomposition of physical amplitudes is completely equivalent to the application of the Wigner-Eckart theorem for the group SU(3). One obtains the amplitudes for the decays of physical particles into reduced SU(3) amplitudes, and the connection between these two bases are simply Clebsch-Gordan coefficients. There are two ways to achieve such a decomposition.

The first method is to work with roots, weights, and ladder operators in the usual manner of Wigner to obtain the desired coefficients. Tables of SU(3) Clebsch-Gordan coefficients for smaller irreducible representations have existed for some time [5], although tables containing all the representations one requires can be more difficult to find [6]. Even with the coefficients in hand, one must convolve several layers of Clebsch-Gordan coefficients to complete this task; this follows because combining two representations is a binary operation, and each additional initial- and final-state particle requires another product. One must also take care to observe the proper phase conventions, which ultimately arise when one requires representations and their conjugates to obey simultaneously the same phase convention for ladder operators.

The second method is to work directly with tensors. Indeed, Clebsch-Gordan coefficients are simply the coefficients of the couplings of tensors that have been appropriately symmetrized, normalized, and rendered traceless. The appeal of this approach is that one can immediately think of the tensors as pieces of the interaction Hamiltonian. In any case, both methods must give identical results, and we have confirmed this through direct calculation.

In either approach, the phase differences between representations and their conjugates must be included in some fashion. These phases arise from the convention one adopts in relating physical states to weights in group space. The most convenient means of doing so is to take the fundamental and fundamental conjugate representations to consist of the quark flavor states

$$\mathbf{3} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad \bar{\mathbf{3}} = \begin{pmatrix} \bar{d} \\ -\bar{u} \\ \bar{s} \end{pmatrix}. \tag{2.1}$$

Including an additional sign for each $\bar{u}$ permits one to assign the physical mesons to the weights in SU(3) representations without additional phases. In this convention, the flavor wavefunctions of the mesons of interest are:

$$K^+ = +u\bar{s}, \qquad K^0 = +d\bar{s}, \tag{2.2}$$

$$\pi^+ = +u\bar{d}, \qquad \pi^0 = -\frac{1}{\sqrt{2}}(+u\bar{u} - d\bar{d}), \qquad \pi^- = -d\bar{u}, \tag{2.3}$$

$$\eta_8 = -\frac{1}{\sqrt{6}}(+u\bar{u} + d\bar{d} - 2s\bar{s}), \tag{2.4}$$

$$\bar{K}^0 = +s\bar{d}, \qquad K^- = -s\bar{u}, \tag{2.5}$$

$$\eta_1 = -\frac{1}{\sqrt{3}}(+u\bar{u} + d\bar{d} + s\bar{s}), \tag{2.6}$$

for the light meson nonet P,

$$B_u^+ = +\bar{b}u, \qquad B_d^0 = +\bar{b}d, \qquad B_s^0 = +\bar{b}s, \tag{2.7}$$

for the triplet B's,

$$\bar{D}^0 = +\bar{c}u, \qquad D^- = +\bar{c}d, \qquad D_s^- = +\bar{c}s, \tag{2.8}$$

for the triplet D's, and

$$D^0 = -c\bar{u}, \qquad D^+ = +c\bar{d}, \qquad D_s^+ = +c\bar{s}, \tag{2.9}$$

for the antitriplet D's. In addition, there is the charmonium singlet

$$\eta_c = +\bar{c}c. \tag{2.10}$$

The physical mesons η, η' are defined through the SO(2) rotation

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} -\cos\theta & +\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_1 \end{pmatrix}, \tag{2.11}$$

where the peculiar sign conventions on the mixing are defined so that the angle θ agrees with that of Gilman and Kauffmann [7], in which the mixing is phenomenologically determined to assume a value of $\theta \simeq -20^\circ$.

In the tensor approach the signs from the above phase convention are ignored. Indeed, the light meson pseudoscalar octet is taken to be represented by the traceless matrix

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta_8 \end{pmatrix}, \tag{2.12}$$

so that the signs of π^0, π^-, η_8, and K^- are opposite to those in Eqs. (2.2)–(2.6). Similarly, in a direct tensor approach one drops the above signs for both η_1 and D^0. These differences in convention result in different signs in the physical amplitudes; there is one relative sign change for each time one of the aforementioned particles appears in an amplitude. These differences are trivial to implement.

III. COUNTING AMPLITUDES

In completely broken flavor SU(3) for a sector of given quantum numbers (corresponding to a Hamiltonian with particular eigenvalues of the diagonal generators taken to be the isospin third component I_3 and the hypercharge Y) there must be exactly as many reduced SU(3) amplitudes as distinct physical processes. This is just a statement of the completeness of the amplitude basis in either physical or group-theoretical terms. For example, consider the case of $B \to PP$, where P here and throughout the paper designates the light pseudoscalar nonet consisting of π, K, η, η' (the octet and singlet components of the physical

η, η' are designated η_8 and η_1, respectively, and are related by Eq. (2.11)). Because group theory relates processes with the same Hamiltonian ΔI_3 and ΔY (which are equivalent to electric charge ΔQ and strangeness ΔS changes[1]), and in any process electric charge is conserved, sets of amplitudes with a particular strangeness change ΔS are related by group theory. For processes not changing strangeness between initial and final states, for example, one counts twelve amplitudes when both P's are octet mesons, four with one octet P and the other being the singlet η_1, and one ($B_d^0 \to \eta_1 \eta_1$) with both P's being singlets. It must be that there are exactly equal numbers of SU(3) reduced amplitudes for each set of particle representations, and this is indeed the case (see Appendix).

In order to count these amplitudes, one must construct the most general possible transformation structure for the interaction Hamiltonian. Because the Hamiltonian connects initial to final states via the matrix elements $\langle f | \mathcal{H} | i \rangle$, the most general interaction Hamiltonian consists of exactly those representations $\mathbf{R}$ appearing in $\mathbf{f} \otimes \bar{\mathbf{i}}$. The labels i and f here are used to denote both states and SU(3) representations. There is one further complication in that states of SU(3) representations are uniquely distinguished when eigenvalues of not only I_3 and Y but also the isospin Casimir I^2 are specified. The full reduced SU(3) amplitude is thus described by the notation $\langle f || R_I || i \rangle$.

If the two final-state particles are both in the same SU(3) representation $\mathbf{f}'$, then their amplitudes obey one further restriction owing to the Pauli Exclusion Principle. Because the initial and both final particles are spin-zero, the spatial part of the final wavefunction is s-wave and therefore symmetric under interchange of particle labels. But because the final-

---

[1]The exact relations are $Q = I_3 + \frac{1}{2}Y + Q_{\mathrm{h}}$ and $S = Y - \frac{1}{3}T$, where Q_{h} is the charge of quarks not belonging to the SU(3) flavor triplet, and T is the triality of the SU(3) representation, which is the number of fundamental representation indices modulo 3 required to build a tensor transforming under the given representation. For $\mathbf{3}$,$\bar{\mathbf{3}}$, this number is ± 1, whereas for octets and singlets it is zero.

state bosons transform under the same representation of SU(3), they are identical particles modulo SU(3) indices, and the total wavefunction must be symmetric under exchanging the two particles. Therefore, the flavor wavefunction alone must also be symmetric under this interchange. This symmetrization, which we write as $\mathbf{f} = (\mathbf{f}' \otimes \mathbf{f}')_S$, eliminates a number of possible representations. For example, in the case of $B \to PP$ with P in the octet, the amplitudes *a priori* transform as

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8}_S \oplus \mathbf{8}_A \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}, \tag{3.1}$$

but the only ones allowed by the Exclusion Principle are

$$(\mathbf{8} \otimes \mathbf{8})_S = \mathbf{1} \oplus \mathbf{8}_S \oplus \mathbf{27}. \tag{3.2}$$

Note that this restriction fails to hold if the final state does not possess a completely symmetric spatial wavefunction, as is the case for arbitrary initial and final spin states. It is also required that the final state particles occupy two weights in the same representation $\mathbf{f}'$, not merely two distinct copies of $\mathbf{f}'$.

It may seem odd that what we call the reduced matrix element $\langle f || R_I || i \rangle$ is dependent upon the isospin Casimir I, not just the SU(3) irreducible representation, of the Hamiltonian operator. This seems to contradict the Wigner-Eckart theorem, which states that *all* of the matrix elements of a particular tensor operator, for states in given initial and final state representations, are related by Clebsch-Gordan coefficients. While the theorem is certainly true for each tensor operator contributing to a physical process, the Hamiltonian itself may have dynamical coefficients that are unequal for different components of a given representation. To be explicit, let us adopt the notation for SU(3) Clebsch-Gordan coefficients of de Swart [5]. The coefficient coupling the representations $\mathbf{R_a} \otimes \mathbf{R_b} \to \mathbf{R_c}$ is indicated by

$$\left(\begin{array}{ccc} R_a & R_b & R_c \\ I_a I_{a3} Y_a & I_b I_{b3} Y_b & I_c I_{c3} Y_c \end{array} \right) . \tag{3.3}$$

For brevity, let us denote the quantum numbers I, I_3, Y within an SU(3) representation by the collective label ν. Then the physical amplitude $\mathcal{A}$ is decomposed into our reduced matrix elements by

$$\mathcal{A}(i^{R_c}_{\nu_c} \to f^{R_a}_{\nu_a} f^{R_b}_{\nu_b}) = (-1)^{\left(I_3+\frac{Y}{2}+\frac{T}{3}\right)_{\overline{R}_c}} \sum_{\substack{R',\nu' \\ R,\nu}} \left(\begin{array}{ccc} R_a & R_b & R' \\ \nu_a & \nu_b & \nu' \end{array}\right) \left(\begin{array}{ccc} R' & \overline{R}_c & R \\ \nu' & -\nu_c & \nu \end{array}\right) \langle R' || R_\nu || R_c \rangle \,. \quad (3.4)$$

Note the order of coupling of the representations: First, the final-state representations are coupled via $\mathbf{R_a} \otimes \mathbf{R_b} \to \mathbf{R}'$, and this representation in turn is coupled to the Hamiltonian through the conjugate of the initial representation, $\mathbf{R}' \otimes \overline{\mathbf{R}}_\mathbf{c} \to \mathbf{R}$, or $\mathbf{f} \otimes \bar{\mathbf{i}} \to \mathcal{H}$. As pointed out in the Appendix, coupling in this order ensures that the Clebsch-Gordan matrices are orthogonal. The phase in the above expression arises from the fact that we use not the initial representation $\mathbf{i}$, but its conjugate $\bar{\mathbf{i}}$; T is the triality of the representation $\overline{\mathbf{R}}_\mathbf{c}$, as defined in Sec. 2, and guarantees the reality of the phase. In the present case, it induces an additional sign on decays of B_u^+ but not B_d^0 or B_s^0. On the other hand, we may choose to couple representations in a more standard order. If we decompose the Hamiltonian as

$$\mathcal{H} = \sum_{R,\nu} c_{R,\nu} \mathcal{H}^R_\nu , \quad (3.5)$$

then the expression for the physical amplitude becomes

$$\mathcal{A}(i^{R_c}_{\nu_c} \to f^{R_a}_{\nu_a} f^{R_b}_{\nu_b}) = \sum_{R,\nu} c_{R,\nu} \sum_{R',\nu'} \left(\begin{array}{ccc} R_a & R_b & R' \\ \nu_a & \nu_b & \nu' \end{array}\right) \left(\begin{array}{ccc} R & R_c & R' \\ \nu & \nu_c & \nu' \end{array}\right) \langle R' || R || R_c \rangle \,, \quad (3.6)$$

which produces the usual Wigner-Eckart reduced amplitude $\langle R' || R || R_c \rangle$, independent of any other quantum numbers, but the Hamiltonian coefficients c have nontrivial ν dependence in general. Note that the products performed here are $\mathbf{R_a} \otimes \mathbf{R_b} \to \mathbf{R}'$ and $\mathbf{R} \otimes \mathbf{R_c} \to \mathbf{R}'$, or $\mathcal{H} \otimes \mathbf{i} \to \mathbf{f}$. The relation between the two reduced amplitudes can be established by the use of symmetry and completeness relations satisfied by the SU(3) Clebsch-Gordan coefficients [5]. Up to a phase dependent upon the representations coupled,

$$\langle R' || R_\nu || R_c \rangle = c_{R,\nu} \sqrt{\frac{\dim R'}{\dim R}} \, \langle R' || R || R_c \rangle \,. \quad (3.7)$$

From this expression we see that the two definitions differ in that ours absorbs the dynamical coefficient appearing with the given operator in the Hamiltonian. Once one specifies, as in the Appendix, the values of I_3 and Y for the Hamiltonian operator, the only free index

remaining in ν is I. Finally, in the case that the two final-state particles are in the same representation as described above so that $R_a = R_b$, then in the above expressions one makes the symmetrizing substitution

$$\begin{pmatrix} R_a & R_b & R' \\ \nu_a & \nu_b & \nu' \end{pmatrix} \to \frac{1}{\sqrt{2}} \left[\begin{pmatrix} R_a & R_b & R' \\ \nu_a & \nu_b & \nu' \end{pmatrix} + \begin{pmatrix} R_b & R_a & R' \\ \nu_b & \nu_a & \nu' \end{pmatrix} \right] . \tag{3.8}$$

Note that, for identical final-state particles ($\nu_a = \nu_b$), this substitution induces an additional factor of $\sqrt{2}$ in the amplitude. While this factor complicates what we mean by the physical amplitude, the substitution Eq. (3.8) nevertheless preserves the orthogonality of the Clebsch–Gordan matrices, as can be shown by either symbolic or direct numerical calculation. A trivial example of this factor is illustrated by the analogous case of spin SU(2), where starting with the four two-spin basis states $\uparrow\uparrow$, $\uparrow\downarrow$, $\downarrow\uparrow$, $\downarrow\downarrow$, the symmetrization indicated by Eq. (3.8) correctly gives $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$, but also $\frac{1}{\sqrt{2}}(2\uparrow\uparrow)$ and $\frac{1}{\sqrt{2}}(2\downarrow\downarrow)$.

We close this section by explaining the physical interpretation of the amplitudes computed in the Appendix. Whereas the physical amplitude for distinct spinless final-state particles ($\nu_a \neq \nu_b$) simply squares to the measurable rate, one must include additional Bose symmetry factors when the final-state particles are identical ($R_a = R_b$ and $\nu_a = \nu_b$). We have noted that the amplitude given by Eq. (3.4) with the substitution Eq. (3.8) for identical particles is a factor $\sqrt{2}$ larger than the naive definition of the physical amplitude. On the other hand, to obtain the decay rate one multiplies the naive amplitude by an exchange factor 2! (giving the usual physical amplitude), squares, and divides by an identical particle factor 2! in the rate to avoid multiple counting. It follows that the rate for $\nu_a = \nu_b$ is twice the naive amplitude squared, or simply the amplitude of Eq. (3.4) squared, the same as for $\nu_a \neq \nu_b$. Because the universal rule *rate* = *amplitude*^{2} is simpler than keeping track of factors of two in certain cases, we present in the Appendix the amplitudes given by Eq. (3.4), using Eq. (3.8) in all cases where $R_a = R_b$.

IV. AMPLITUDE RELATIONS

The above counting describes how one enumerates the complete set of amplitudes for arbitrarily broken SU(3). This counting holds even is there is no good physical reason to organize particles into SU(3) multiplets. For example, one could take eight arbitrary particles and call them an octet of SU(3), and the group-theoretical decomposition, which is purely mathematical, would remain true. Clearly one requires physical input to make practical use of the group theory. If one finds a physical reason why a particular SU(3) reduced amplitude should vanish, then the particular combination of physical amplitudes to which it is equal also vanishes. This corresponds to taking the inverse of the transformation matrix $\mathcal{O}$, for which each row decomposes a particular physical amplitude into reduced SU(3) amplitudes. However, as noted in the Appendix, the SU(3) reduced matrix elements are normalized so that the basis transformation matrices $\mathcal{O}$ are orthogonal, hence $\mathcal{O}^{-1} = \mathcal{O}^T$, and the relation associated with the vanishing of a particular SU(3) reduced amplitude is obtained by merely reading off the entries from the corresponding column of $\mathcal{O}$. The relations also hold for the charge-conjugated states, although the presence of CP violation means that amplitudes for individual processes do not necessarily equal the amplitudes for their conjugate processes. Finally, as discussed in Sec. 2, these matrices are obtained using a particular phase convention. Choosing another either results in changing the signs of particular particle states in terms of their quark-antiquark indices or changing the signs in the definition of reduced matrix elements. These correspond respectively to changing the signs of rows or columns of the matrices in the Appendix, operations which do not affect their orthogonality.

One begins by assuming a form for the unbroken Hamiltonian. For the case of B decay, this is of course the four-quark Hamiltonian derived from the tree-level weak interaction

$$\mathcal{H}_{\rm int} = \frac{4G_F}{\sqrt{2}} V^*_{q_1 b} V_{q_2 q_3} (\bar{b}_L \gamma^\mu q_{1L})(\overline{q_2}_L \gamma_\mu q_{3L}), \tag{4.1}$$

where $q_{1,2}$ are charge $+2/3$ (u,c) quarks and q_3 is a charge $-1/3$ (d,s) quark. Note that there are several physical assumptions already included in this ansatz. In writing the Hamiltonian

this way, the physical B decay is assumed to be dominated by the decay of the $\bar{b}$ quark into a $\bar{c}$ or $\bar{u}$ quark with the emission of a virtual W^-, which subsequently decays into a quark-antiquark pair. All other contributions involving QCD renormalization effects or penguins, for example, are considered negligible in this limit.

One can analyze the Hamiltonian for each case of flavor content. The field operators q_i, $\bar{q}_i$ for $q_i = u, d, s$ transform as components of $\bar{\mathbf{3}}$, $\mathbf{3}$ respectively, because these operators respectively destroy initial-state quarks and antiquarks; $q_i = c$ is of course a singlet in flavor SU(3). For the case $\Delta C = 0$ with no $c\bar{c}$ pair in the final state ($B \to PP$), the SU(3) representations allowed by $\mathcal{H}_{\rm int}$ are those in

$$\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \bar{\mathbf{3}} \oplus \mathbf{6} \oplus \overline{\mathbf{15}}. \tag{4.2}$$

The redundancy of the $\bar{\mathbf{3}}$ representation in the Hamiltonian is irrelevant, because one cannot distinguish in this case the two contributions, which transform in the same way. It follows that the lowest-order Hamiltonian has pieces transforming as $\bar{\mathbf{3}}$, $\mathbf{6}$, and $\overline{\mathbf{15}}$, but not $\mathbf{24}$ or $\overline{\mathbf{42}}$ (see Eqs. (A2), (A6)), and so for either $\Delta S = 0$ or $\Delta S = +1$ there are five amplitude relations corresponding to the five vanishing amplitude combinations transforming under $\mathbf{24}$ or $\overline{\mathbf{42}}$ in Eqs. (A2), (A6). It is readily seen why the $\mathbf{24}$ and $\overline{\mathbf{42}}$ representations do not appear at leading order: These representations require three indices in either the fundamental or fundamental conjugate representation (see Eq. (A1)), and this is impossible with a four-quark (two-quark, two-antiquark) operator. These relations may be obtained as the amplitude combinations that obtain no contributions from the operators analyzed in Refs. [3].

In order to analyze isospin content, we must further specify the number of strange quarks created or destroyed in the process. For $B \to PP$ with $\Delta S = 0$, the possible isospins are those in $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$, namely $I = \frac{1}{2}, \frac{3}{2}$; and for $\Delta S = +1$ only two quarks are light, so $I = 0, 1$ are possible. In the SU(3) symmetry limit, this does not eliminate any additional amplitudes besides the ones mentioned above. The analysis for other values of ΔS in $B \to PP$ is straightforward: More s or $\bar{s}$ quarks implies that the maximum allowed I from a four-quark Hamiltonian is smaller.

Let us now consider SU(3)-breaking corrections to the lowest-order Hamiltonian. The simplest such breaking originates through insertions of the strange quark mass,

$$\mathcal{H}_s = m_s \bar{s} s, \tag{4.3}$$

which transforms as an $I = 0$, $Y = 0$ octet plus singlet in SU(3). Clearly neither piece changes the isospin of the Hamiltonian; this would be accomplished by insertions of the up or down masses, which are much smaller. Let us consider SU(3) breaking linear in m_s. In the case of $B \to PP$, the Hamiltonian contains pieces transforming under

$$(\bar{\mathbf{3}} \oplus \mathbf{6} \oplus \overline{\mathbf{15}}) \otimes (\mathbf{1} \oplus \mathbf{8}) = \bar{\mathbf{3}} \oplus \mathbf{6} \oplus \overline{\mathbf{15}} \oplus \mathbf{24} \oplus \overline{\mathbf{42}}, \tag{4.4}$$

not counting multiplicities. Comparing to Eqs. (A2) and (A6), we see that every allowed Hamiltonian SU(3) representation occurs. Nothing is gained group-theoretically by stopping at linear order in strange-quark masses. This interesting conclusion turns out to be true for every decay considered in the Appendix, when all allowed values of ΔS are considered. It is generally true for an amplitude with a total of three mesons (each of which is group-theoretically a quark-antiquark state) between the initial and final states, because the Hamiltonian is a six-quark (three-quark, three-antiquark) operator, and so every representation that can connect the initial and final states can occur in the Hamiltonian. On the other hand, because $\bar{s}s$ is an $I = 0$ operator, the restriction that only $I = \frac{1}{2}, \frac{3}{2}$ are allowed for $\Delta S = 0$ and $I = 0, 1$ are allowed for $\Delta S = +1$ remains true even when the lowest-order Hamiltonian is corrected with an arbitrary number of $\bar{s}s$ insertions. One then still has the relations

$$\left\langle 27 || \overline{42}_{I=\frac{5}{2}} || 3 \right\rangle = 0 \;\; (\text{for } \Delta S = 0), \tag{4.5}$$

$$\left\langle 27 || \overline{42}_{I=2} || 3 \right\rangle = 0 \;\; (\text{for } \Delta S = +1). \tag{4.6}$$

These relations follow entirely from the fact that we have broken the SU(3) symmetry in the Hamiltonian, but not isospin.

A similar analysis holds for Hamiltonians with different flavor contents. In general, the group theory is simpler for decays with charm quarks in the final state, since charm transforms as an SU(3) singlet.

Amplitude relations may also appear for particular values of ΔS since entire SU(3) representations may be disallowed by the particular Hamiltonian used. For example, in the case $B \to D\bar{D}$ or $B \to \eta_c P$ with $\Delta S = +1$ (Eqs. (A33), (A38)), the lowest-order tree-level Hamiltonian is

$$\mathcal{H}_0 \sim (\bar{b}c)(\bar{c}s), \tag{4.7}$$

where we have suppressed all except flavor indices. In terms of flavor SU(3), this operator falls into the $\bar{\mathbf{3}}$ representation. If we include one insertion of SU(3) breaking of the form Eq. (4.3), then

$$\delta\mathcal{H} \sim (\bar{b}c)(\bar{c}s)(\bar{s}s). \tag{4.8}$$

Group-theoretically, one may first combine $(s \otimes s) \in \bar{\mathbf{6}} \oplus \mathbf{3}$. However, $\mathbf{3}$ does not contain a state with the hypercharge $Y = +2/3$ of $(s \otimes s)$ and so does not occur. Taking the product of these representations with the remaining $\bar{s}$,

$$\bar{\mathbf{6}} \otimes \mathbf{3} = \overline{\mathbf{15}} \oplus \bar{\mathbf{3}}, \qquad \mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}. \tag{4.9}$$

In particular, a Hamiltonian $\mathbf{6}$, which is allowed if all values of ΔS are considered, does not occur in the case $\Delta S = +1$, and so one obtains an amplitude relation,

$$\langle 8 || 6_{I=1} || 3 \rangle = 0. \tag{4.10}$$

Similar reasoning applies to the decays $B \to \bar{D}P$ with $\Delta S = +1$ (Eq. (A27)), for which one finds

$$\left\langle 15 || 10_{I=\frac{3}{2}} || 3 \right\rangle = 0, \tag{4.11}$$

for tree-level plus first-order SU(3) symmetry-breaking terms in the Hamiltonian. Note that this conclusion is specific to the choice of form for the Hamiltonian. It should also be noted that these relations may be obtained through the isospin analysis described above.

In greatest generality, the SU(3) reduced matrix elements are all independent, because the physical amplitudes need not be related in any way. Each such reduced matrix element corresponds to a component of the Hamiltonian transforming under a particular representation of SU(3) with a particular value of isospin, and so in the most general situation the coefficient of each component is independent. However, in the usual case, one uses a Hamiltonian with particular operators (typically written in terms of quark fields) that can be explicitly decomposed under SU(3). Then the reduced matrix elements allowed by these operators are related by a product of the Clebsch-Gordan coefficients obtained by projecting the given operators onto SU(3) representations multiplied by the explicit coefficients of the original operators themselves. As an example, consider the $B \to PP$ reduced matrix elements $\left\langle f || \overline{15}_{I=1} || i \right\rangle$ (in $\Delta S = 0$) and $\left\langle f || \overline{15}_{I=\frac{3}{2}} || i \right\rangle$ (in $\Delta S = +1$). A tree-level analysis suggests that these matrix elements are dominated by the Hamiltonian

$$\mathcal{H}_{\rm int} = \frac{4G_F}{\sqrt{2}} \left[V_{ub}^* V_{ud} \left(\bar{b}_L \gamma^\mu u_L \right) \left(\bar{u}_L \gamma_\mu d_L \right) + V_{ub}^* V_{us} \left(\bar{b}_L \gamma^\mu u_L \right) \left(\bar{u}_L \gamma_\mu s_L \right) \right] . \qquad (4.12)$$

The gluonic penguin diagram does not contribute to the part of the Hamiltonian transforming as a $\overline{\mathbf{15}}$ when SU(3) is unbroken since the gluon is an SU(3) singlet, and so too must be the quark-antiquark pair produced by it. Thus the penguin process has flavor content only through the decays $\bar{b} \to \bar{d}$ ($I = \frac{1}{2}$) or $\bar{b} \to \bar{s}$ ($I = 0$), which transform as components of a $\bar{\mathbf{3}}$. The SU(3) structure of the four-quark operators in Eq. (4.12) is manifest, and one can immediately project them onto the $\overline{\mathbf{15}}$ to obtain the corresponding Clebsch-Gordan coefficients. The two Hamiltonian operators otherwise differ only in their coefficients, two combinations of CKM elements. The result of the calculation is

$$\frac{\left\langle f || \overline{15}_{I=1} || i \right\rangle}{\left\langle f || \overline{15}_{I=\frac{3}{2}} || i \right\rangle} = + \frac{\sqrt{3}}{2} \frac{V_{us}}{V_{ud}}, \qquad (4.13)$$

regardless of the initial- or final-state representations (i,f) of the particles. The corresponding expression within a given strangeness sector is even simpler, because then the CKM element for each $\overline{\mathbf{15}}$ component is the same. In particular,

$$\frac{\left\langle f || \overline{15}_{I=\frac{1}{2}} || i \right\rangle}{\left\langle f || \overline{15}_{I=\frac{3}{2}} || i \right\rangle} = + \frac{1}{2\sqrt{2}}, \qquad \frac{\left\langle f || \overline{15}_{I=0} || i \right\rangle}{\left\langle f || \overline{15}_{I=1} || i \right\rangle} = + \frac{1}{\sqrt{2}}, \qquad (4.14)$$

for $\Delta S = 0$ and $\Delta S = +1$, respectively. The key to the simplicity of the ratios in this example is that only one operator structure (the four-quark operator) dominates the Hamiltonian for the given decays; when several different operator structures contribute to the Hamiltonian for a particular decay, ratios like Eqs. (4.13), (4.14) are replaced by

$$\frac{\langle f||R_I||i\rangle}{\langle f||R'_{I'}||i\rangle} = \frac{\sum_j c_j \,\mathcal{C}_j}{\sum_k c_k \,\mathcal{C}'_k}, \tag{4.15}$$

where $\mathcal{C}$ and $\mathcal{C}'$ are Clebsch-Gordan coefficients, and c are coefficients of the different components of the Hamiltonian. This is the case when, for example, penguin diagrams, diagrams involving the participation of the spectator quark, or SU(3) corrections are significant. For example, in the case of the $\overline{\mathbf{15}}$, $\mathcal{O}(m_s)$ corrections to the Hamiltonian (4.12) introduce an additional operator transforming as a $\overline{\mathbf{15}}$, since in Eq. (4.2) $\overline{\mathbf{15}} \otimes \mathbf{8} \supset \overline{\mathbf{15}}$. On the other hand, even lowest-order Hamiltonian operators transforming under $\bar{\mathbf{3}}$ or $\mathbf{6}$ corrected by the SU(3) octet breaking produce $\overline{\mathbf{15}}$'s. Then the relations (4.13), (4.14) are replaced with ones of the form (4.15). For the $\overline{\mathbf{15}}$ such corrections may be small, but it is often the case that two or more operators of the same numerical order and distinct coefficients appear when considering a particular Hamiltonian representation; in such cases, Eq. (4.15) must be used.

We comment briefly on the relations that can be derived through the above analysis and those derived in Refs. [2]. Each relation unbroken by the full set of quark interactions described in [2] appears because the quark diagrams may be re-interpreted as collections of quark fields with definite SU(3) flavor properties in a Hamiltonian, and as such give rise to a set of SU(3) irreducible representations. In this way, the quark diagram approach is group-theoretically equivalent to the more formal approach, as was first pointed out by Zeppenfeld [3]. All possible representations that do not appear in the Hamiltonian thus give rise to amplitude relations. The analysis of isospin relations is particularly straightforward, as we have discussed. Indeed, in addition to the numerous isospin relations derived in [2], we add one they have omitted,

$$\mathcal{A}(B_s^0 \to \eta_c \pi^0) = 0, \tag{4.16}$$

which vanishes because neither the Hamiltonian Eq. (4.7) nor the SU(3)-breaking correction (4.8) contains an $I = 1$ piece, while the process in question is pure $I = 1$ (only the pion carries isospin in the decay). The latter fact is corroborated by a quick glance at the third row of Eq. (A38). The SU(3) relations in [2] may be obtained through analysis like that leading to Eqs. (4.13)–(4.14), whereas their more detailed analysis of neglecting certain quark diagrams would be obtained in our language by decomposing the corresponding Hamiltonian quark operator into a combination of different SU(3) representations. Finally, a number of η, η' relations are possible when one includes a value for the mixing of η_1, η_8 as discussed in Sec. 2.

V. PROSPECTS AND IMPROVEMENTS

The purpose of the discussion and formulas contained in this work is to provide a complete analysis of the group-theoretical problem of the decay of B mesons into two pseudoscalar mesons. Because it is completely general in the mathematical sense, it provides a valuable tool of analysis for researchers performing computations of such processes. But this strength is also its weakness: No physical content was included, except for trivial illustrative examples. Nevertheless, once any particular Hamiltonian is adopted, one can immediately see exactly which physical amplitudes vanish or are related for symmetry reasons, and why.

The fully consistent application of this analysis to the physical problem of unambiguously extracting CKM elements and strong-interaction final-state phase shifts requires one to impose an SU(3) decomposition on the full Hamiltonian taking part in the B decay, including short-distance QCD corrections and SU(3) symmetry breaking. Because gluons transform as singlets under flavor SU(3), many extremely complicated diagrams involving large numbers of gluons and sea quark-antiquark pairs can still be taken into account in a simple way using the flavor symmetry. On the other hand, complications may arise, such as the presence of potentially important electroweak penguin diagrams [8], which have a nontrivial flavor structure and thus must be treated carefully. By choosing reduced matrix

elements insensitive to these corrections, one may be able to avoid this difficulty. Amplitude relations that survive these corrections will be invaluable for studying the detailed structure of the CKM matrix.

There is much to be learned even from the amplitudes that are not suppressed. In this case, the interesting question is whether the reduced matrix elements obey the numerical hierarchy predicted by a more naive analysis, based upon some physical model, that estimates the size of operator coefficients. With enough rates eventually measured, one will be able to compute reduced matrix elements directly, without recourse to model-dependent assumptions. For example, the issue of whether gluonic penguins or tree-level amplitudes dominate a given process will be directly resolved, inasmuch as their corresponding Hamiltonians transform differently under SU(3). This knowledge can be applied to many other interactions involving heavy quarks. We plan to address these questions in greater detail in a future publication.

Acknowledgments

This work is supported in part by the Department of Energy under contract DOE-FG03-90ER40546. The research of B. G. is also funded in part by the Alfred P. Sloan Foundation.

REFERENCES

[1] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963);
M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[2] M. Gronau, O. F. Hernández, D. London, and J. L. Rosner, Phys. Rev. D **50**, 4529 (1994); Phys. Lett. B **333**, 500 (1994); Phys. Rev. D **52**, 6356 (1995); Phys. Rev. D **52**, 6374 (1995);
M. Gronau, D. London, and J. L. Rosner, Phys. Rev. Lett. **73**, 21 (1994);
M. Gronau and J. L. Rosner, Report No. Technion-PH-95-26, EFI-95-59 [hep-ph/9509325] (unpublished); Report No. Technion-PH-95-28, EFI-95-66 [hep-ph/9510363] (unpublished);
A. S. Dighe, Report No. EFI-95-52 [hep-ph/9509287] (unpublished);
A. S. Dighe, M. Gronau, and J. L. Rosner, Report No. Technion-PH-95-27, EFI-95-60 [hep-ph/9509428] (unpublished).

[3] D. Zeppenfeld, Z. Phys. C **8**, 77 (1981);
M. J. Savage and M. B. Wise, Phys. Rev. D **39**, 3346 (1989); **40**, 3127(E) (1989).

[4] B. Grinstein and R. F. Lebed, in preparation.

[5] J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963).

[6] T. A. Kaeding, Lawrence Berkeley Laboratory Report No. LBL-36659 and University of California Report No. UCB-PTH-95/01 [nucl-th/9502037] (unpublished).

[7] F. J. Gilman and R. Kauffmann, Phys. Rev. D **36**, 2761 (1987).

[8] N. G. Deshpande and X.-G. He, Phys. Rev. Lett. **74**, 26 (1995); *ibid.*, 4099(E) (1995);
N. G. Deshpande, X.-G. He, and S. Oh, University of Oregon Report No. OITS-593 [hep-ph/9511462] (unpublished).

[9] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, England, 1935).

APPENDIX: CLEBSCH-GORDAN TABLES

Presented below is the complete decomposition in terms of SU(3) reduced amplitudes for decays of a flavor-triplet B meson into a pair of pseudoscalars with $\Delta C = 0$, $+1$, or -1. The phase convention is chosen to agree with that of Condon and Shortley [9] for isospin SU(2), which is defined by two conditions. First, the phase in the definition of the isospin raising and lowering operators acting on a given isospin eigenstate is chosen to be $+1$; this establishes phases within a particular isomultiplet. Second, to establish the relative phase between multiplets (in this case isomultiplets with a common value of hypercharge), one considers the couplings of the two factor representations ($I^{(a)}$, $I^{(b)}$, with $I^{(a)} \geq I^{(b)}$) to a given product representation I. Then, for the state of highest weight in the product multiplet ($I_3 = I$), the coupling

$$\left\langle I^{(a)}\, I_3^{(a)};\, I^{(b)}\, I_3^{(b)} | I\, I \right\rangle$$

is chosen to have phase $+1$ when $I_3^{(a)}$ is the largest such value that a nonzero coupling occurs. The relative phases for states with different values of hypercharge are fixed in the SU(3) convention of de Swart [5], in that the first condition of the Condon-Shortley convention is extended to hold for both isospin and V-spin ($V_3 \equiv \frac{3}{4}Y + \frac{1}{2}I_3$). The arbitrary choice of V-spin instead of U-spin or some combination leads to the convention in Sec. 2 that the fundamental conjugate state $\bar{u}$ transforms with phase opposite to that of $\bar{d}$, $\bar{s}$. The phase convention on the physical states is also described in Sec. 2. Each matrix, as one can check, is orthogonal, thus establishing that the SU(3) reduced amplitudes form an orthonormal basis equivalent to the amplitudes $\mathcal{A}$.

The physical decay rate is obtained in all cases by squaring the quantity $\mathcal{A}$. If the two final-state particles are identical, the quantity $\mathcal{A}$ is the usual physical amplitude (which already includes an identical particle factor) divided by $\sqrt{2}$, as described in Sec. 3.

All SU(3) representations are indicated below merely by their dimensions; this creates no problem for us, since no two distinct representations with the same dimensionality appear

in this analysis. For convenience, we also present here the equivalent weight notation (p, q), where p and q respectively indicate the number of fundamental and fundamental conjugate indices in the tensor representation. The Young tableau then consists of a row of $p+q$ boxes over a row of q boxes. The representation is labeled with a bar if $p < q$.

$$\begin{aligned} \mathbf{1} &= (0,0), & \mathbf{3} &= (1,0), & \mathbf{6} &= (2,0), \\ \mathbf{8} &= (1,1), & \mathbf{10} &= (3,0), & \mathbf{15} &= (2,1), \\ \mathbf{24} &= (3,1), & \mathbf{27} &= (2,2), & \mathbf{42} &= (3,2). \end{aligned} \tag{A1}$$

The expressions below are divided into sections first by particle content of the final state, and then by the number of units of strangeness changed in the B decay.

1. $B \to P, P$

a. $\Delta S = 0$ ($\Delta I_3 = +\frac{1}{2}$, $\Delta Y = -\frac{1}{3}$)

For P, P both octet mesons, denote $\mathbf{u}_{8,8}^{S=0} = \mathcal{O}_{8,8}^{S=0}\, \mathbf{v}_{8,8}^{S=0}$, where

$$\mathbf{u}_{8,8}^{S=0} = \begin{pmatrix} \mathcal{A}(B_u^+ \to K^+\bar{K}^0) \\ \mathcal{A}(B_u^+ \to \pi^+\pi^0) \\ \mathcal{A}(B_u^+ \to \pi^+\eta_8) \\ \mathcal{A}(B_d^0 \to K^+K^-) \\ \mathcal{A}(B_d^0 \to \pi^+\pi^-) \\ \mathcal{A}(B_d^0 \to \pi^0\pi^0) \\ \mathcal{A}(B_d^0 \to \pi^0\eta_8) \\ \mathcal{A}(B_d^0 \to \eta_8\eta_8) \\ \mathcal{A}(B_d^0 \to K^0\bar{K}^0) \\ \mathcal{A}(B_s^0 \to \bar{K}^0\pi^0) \\ \mathcal{A}(B_s^0 \to \bar{K}^0\eta_8) \\ \mathcal{A}(B_s^0 \to K^-\pi^+) \end{pmatrix}, \qquad \mathbf{v}_{8,8}^{S=0} = \begin{pmatrix} \left\langle 1 \| \bar{3}_{I=\frac{1}{2}} \| 3 \right\rangle \\ \left\langle 8 \| \bar{3}_{I=\frac{1}{2}} \| 3 \right\rangle \\ \left\langle 8 \| 6_{I=\frac{1}{2}} \| 3 \right\rangle \\ \left\langle 8 \| \overline{15}_{I=\frac{1}{2}} \| 3 \right\rangle \\ \left\langle 8 \| \overline{15}_{I=\frac{3}{2}} \| 3 \right\rangle \\ \left\langle 27 \| \overline{15}_{I=\frac{1}{2}} \| 3 \right\rangle \\ \left\langle 27 \| \overline{15}_{I=\frac{3}{2}} \| 3 \right\rangle \\ \left\langle 27 \| 24_{I=\frac{1}{2}} \| 3 \right\rangle \\ \left\langle 27 \| 24_{I=\frac{3}{2}} \| 3 \right\rangle \\ \left\langle 27 \| \overline{42}_{I=\frac{1}{2}} \| 3 \right\rangle \\ \left\langle 27 \| \overline{42}_{I=\frac{3}{2}} \| 3 \right\rangle \\ \left\langle 27 \| \overline{42}_{I=\frac{5}{2}} \| 3 \right\rangle \end{pmatrix}, \tag{A2}$$

and $\mathcal{O}_{8,8}^{S=0} =$

$$
\begin{pmatrix}
0 & +\frac{3}{2\sqrt{10}} & +\frac{1}{2}\sqrt{\frac{3}{5}} & +\frac{1}{2\sqrt{10}} & +\frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{15}} & -\frac{1}{3\sqrt{30}} & -\frac{4}{3\sqrt{15}} & +\frac{1}{3\sqrt{3}} & -\frac{2}{3\sqrt{15}} & -\frac{1}{3}\sqrt{\frac{5}{6}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{5}{3}} & 0 & -\frac{1}{\sqrt{6}} & 0 & -\frac{1}{2\sqrt{15}} & -\sqrt{\frac{2}{5}} \\
0 & -\frac{1}{2}\sqrt{\frac{3}{5}} & -\frac{1}{\sqrt{10}} & -\frac{1}{2\sqrt{15}} & -\sqrt{\frac{2}{15}} & -\frac{2}{3}\sqrt{\frac{2}{5}} & -\frac{1}{6\sqrt{5}} & -\frac{2}{3}\sqrt{\frac{2}{5}} & +\frac{1}{3\sqrt{2}} & -\frac{1}{3}\sqrt{\frac{2}{5}} & -\frac{\sqrt{5}}{6} & 0 \\
+\frac{1}{2} & +\frac{1}{\sqrt{10}} & 0 & +\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{5}} & -\frac{1}{6\sqrt{15}} & +\frac{1}{3\sqrt{30}} & -\frac{1}{3}\sqrt{\frac{5}{3}} & -\frac{1}{3\sqrt{3}} & +\frac{2}{3\sqrt{15}} & +\frac{1}{3}\sqrt{\frac{5}{6}} & 0 \\
+\frac{1}{2} & -\frac{1}{2\sqrt{10}} & +\frac{1}{2}\sqrt{\frac{3}{5}} & -\frac{3}{2\sqrt{10}} & 0 & -\frac{1}{6\sqrt{15}} & -\frac{1}{3}\sqrt{\frac{5}{6}} & +\frac{1}{3\sqrt{15}} & -\frac{1}{3\sqrt{3}} & -\frac{1}{3\sqrt{15}} & -\frac{1}{3\sqrt{30}} & +\frac{1}{\sqrt{5}} \\
-\frac{1}{2\sqrt{2}} & +\frac{1}{4\sqrt{5}} & -\frac{1}{2}\sqrt{\frac{3}{10}} & +\frac{3}{4\sqrt{5}} & 0 & +\frac{1}{6\sqrt{30}} & -\frac{1}{3}\sqrt{\frac{5}{3}} & -\frac{1}{3\sqrt{30}} & -\frac{1}{3}\sqrt{\frac{2}{3}} & +\frac{1}{3\sqrt{30}} & -\frac{1}{3\sqrt{15}} & +\sqrt{\frac{2}{5}} \\
0 & -\frac{1}{2}\sqrt{\frac{3}{10}} & -\frac{1}{2\sqrt{5}} & -\frac{1}{2\sqrt{30}} & +\frac{2}{\sqrt{15}} & -\frac{2}{3\sqrt{5}} & +\frac{1}{3\sqrt{10}} & -\frac{2}{3\sqrt{5}} & -\frac{1}{3} & -\frac{1}{3\sqrt{5}} & +\frac{1}{3}\sqrt{\frac{5}{2}} & 0 \\
-\frac{1}{2\sqrt{2}} & -\frac{1}{4\sqrt{5}} & +\frac{1}{2}\sqrt{\frac{3}{10}} & -\frac{3}{4\sqrt{5}} & 0 & +\frac{1}{2}\sqrt{\frac{3}{10}} & 0 & -\sqrt{\frac{3}{10}} & 0 & +\sqrt{\frac{3}{10}} & 0 & 0 \\
-\frac{1}{2} & +\frac{1}{2\sqrt{10}} & +\frac{1}{2}\sqrt{\frac{3}{5}} & -\frac{1}{2\sqrt{10}} & -\frac{1}{\sqrt{5}} & -\frac{7}{6\sqrt{15}} & +\frac{1}{3\sqrt{30}} & +\frac{1}{3\sqrt{15}} & -\frac{1}{3\sqrt{3}} & -\frac{4}{3\sqrt{15}} & +\frac{1}{3}\sqrt{\frac{5}{6}} & 0 \\
0 & +\frac{3}{4\sqrt{5}} & -\frac{1}{2}\sqrt{\frac{3}{10}} & -\frac{3}{4\sqrt{5}} & 0 & +\frac{1}{3}\sqrt{\frac{2}{15}} & -\frac{1}{3}\sqrt{\frac{5}{3}} & -\frac{1}{3\sqrt{30}} & +\frac{2}{3}\sqrt{\frac{2}{3}} & -\frac{1}{3}\sqrt{\frac{2}{15}} & +\frac{1}{3}\sqrt{\frac{5}{3}} & 0 \\
0 & +\frac{1}{4}\sqrt{\frac{3}{5}} & -\frac{1}{2\sqrt{10}} & -\frac{1}{4}\sqrt{\frac{3}{5}} & 0 & -\sqrt{\frac{2}{5}} & 0 & +\frac{1}{\sqrt{10}} & 0 & +\sqrt{\frac{2}{5}} & 0 & 0 \\
0 & -\frac{3}{2\sqrt{10}} & +\frac{1}{2}\sqrt{\frac{3}{5}} & +\frac{3}{2\sqrt{10}} & 0 & -\frac{2}{3\sqrt{15}} & -\frac{1}{3}\sqrt{\frac{5}{6}} & +\frac{1}{3\sqrt{15}} & +\frac{2}{3\sqrt{3}} & +\frac{2}{3\sqrt{15}} & +\frac{1}{3}\sqrt{\frac{5}{6}} & 0
\end{pmatrix},
\tag{A3}
$$

$$
\begin{pmatrix}
\mathcal{A}(B_u^+ \to \pi^+\eta_1) \\
\mathcal{A}(B_d^0 \to \pi^0\eta_1) \\
\mathcal{A}(B_d^0 \to \pi^+\eta_1) \\
\mathcal{A}(B_s^0 \to \pi^0\eta_1)
\end{pmatrix}
=
\begin{pmatrix}
-\frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2} & -\frac{1}{2\sqrt{6}} & -\frac{1}{\sqrt{3}} \\
-\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{4\sqrt{3}} & +\sqrt{\frac{2}{3}} \\
+\frac{1}{4} & -\frac{1}{2}\sqrt{\frac{3}{2}} & +\frac{3}{4} & 0 \\
-\frac{1}{2}\sqrt{\frac{3}{2}} & +\frac{1}{2} & +\frac{1}{2}\sqrt{\frac{3}{2}} & 0
\end{pmatrix}
\begin{pmatrix}
\left\langle 8||\bar{3}_{I=\frac{1}{2}}||3\right\rangle \\
\left\langle 8||\bar{6}_{I=\frac{1}{2}}||3\right\rangle \\
\left\langle 8||\overline{15}_{I=\frac{1}{2}}||3\right\rangle \\
\left\langle 8||\overline{15}_{I=\frac{3}{2}}||3\right\rangle
\end{pmatrix},
\tag{A4}
$$

and

$$
\mathcal{A}(B_d^0 \to \eta_1\eta_1) = (+1)\left\langle 1||\bar{3}_{I=\frac{1}{2}}||3\right\rangle .
\tag{A5}
$$

b. $\Delta S = +1$ ($\Delta I_3 = 0$, $\Delta Y = +\frac{2}{3}$)

For P, P both octet mesons, denote $\mathbf{u}_{8,8}^{S=1} = \mathcal{O}_{8,8}^{S=1}\,\mathbf{v}_{8,8}^{S=1}$, where

$$
\mathbf{u}_{8,8}^{S=1} = \begin{pmatrix} \mathcal{A}(B_u^+ \to K^0\pi^+) \\ \mathcal{A}(B_u^+ \to K^+\pi^0) \\ \mathcal{A}(B_u^+ \to K^+\eta_8) \\ \mathcal{A}(B_d^0 \to K^+\pi^-) \\ \mathcal{A}(B_d^0 \to K^0\pi^0) \\ \mathcal{A}(B_d^0 \to K^0\eta_8) \\ \mathcal{A}(B_s^0 \to \pi^+\pi^-) \\ \mathcal{A}(B_s^0 \to \pi^0\pi^0) \\ \mathcal{A}(B_s^0 \to \pi^0\eta_8) \\ \mathcal{A}(B_s^0 \to \eta_8\eta_8) \\ \mathcal{A}(B_s^0 \to K^+K^-) \\ \mathcal{A}(B_s^0 \to K^0\bar{K}^0) \end{pmatrix}, \qquad \mathbf{v}_{8,8}^{S=1} = \begin{pmatrix} \langle 1||\bar{3}_{I=0}||3\rangle \\ \langle 8||\bar{3}_{I=0}||3\rangle \\ \langle 8||6_{I=1}||3\rangle \\ \left\langle 8||\overline{15}_{I=0}||3\right\rangle \\ \left\langle 8||\overline{15}_{I=1}||3\right\rangle \\ \left\langle 27||\overline{15}_{I=0}||3\right\rangle \\ \left\langle 27||\overline{15}_{I=1}||3\right\rangle \\ \langle 27||24_{I=1}||3\rangle \\ \langle 27||24_{I=2}||3\rangle \\ \left\langle 27||\overline{42}_{I=0}||3\right\rangle \\ \left\langle 27||\overline{42}_{I=1}||3\right\rangle \\ \left\langle 27||\overline{42}_{I=2}||3\right\rangle \end{pmatrix}, \tag{A6}
$$

and $\mathcal{O}_{8,8}^{S=1} =$

$$
\begin{pmatrix}
0 & +\frac{3}{2\sqrt{10}} & -\frac{1}{2}\sqrt{\frac{3}{5}} & +\frac{1}{2}\sqrt{\frac{3}{10}} & +\frac{1}{2}\sqrt{\frac{3}{5}} & +\frac{1}{3\sqrt{5}} & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & +\frac{1}{3\sqrt{2}} & +\frac{1}{3\sqrt{10}} & 0 & -\frac{1}{3} \\
0 & -\frac{3}{4\sqrt{5}} & +\frac{1}{2}\sqrt{\frac{3}{10}} & -\frac{1}{4}\sqrt{\frac{3}{5}} & -\frac{1}{2}\sqrt{\frac{3}{10}} & -\frac{1}{3\sqrt{10}} & -\frac{7}{6\sqrt{5}} & -\frac{1}{3\sqrt{5}} & +\frac{1}{3} & -\frac{1}{6\sqrt{5}} & -\frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\
0 & +\frac{1}{4}\sqrt{\frac{3}{5}} & -\frac{1}{2\sqrt{10}} & +\frac{1}{4\sqrt{5}} & +\frac{1}{2\sqrt{10}} & -\sqrt{\frac{3}{10}} & -\frac{1}{2\sqrt{15}} & +\frac{2}{\sqrt{15}} & 0 & -\frac{1}{2}\sqrt{\frac{3}{5}} & -\frac{1}{\sqrt{6}} & 0 \\
0 & -\frac{3}{2\sqrt{10}} & -\frac{1}{2}\sqrt{\frac{3}{5}} & -\frac{1}{2}\sqrt{\frac{3}{10}} & +\frac{1}{2}\sqrt{\frac{3}{5}} & -\frac{1}{3\sqrt{5}} & -\frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{10}} & 0 & +\frac{1}{3} \\
0 & +\frac{3}{4\sqrt{5}} & +\frac{1}{2}\sqrt{\frac{3}{10}} & +\frac{1}{4}\sqrt{\frac{3}{5}} & -\frac{1}{2}\sqrt{\frac{3}{10}} & +\frac{1}{3\sqrt{10}} & -\frac{7}{6\sqrt{5}} & -\frac{1}{3\sqrt{5}} & -\frac{1}{3} & +\frac{1}{6\sqrt{5}} & -\frac{1}{3\sqrt{2}} & +\frac{\sqrt{2}}{3} \\
0 & +\frac{1}{4}\sqrt{\frac{3}{5}} & +\frac{1}{2\sqrt{10}} & +\frac{1}{4\sqrt{5}} & -\frac{1}{2\sqrt{10}} & -\sqrt{\frac{3}{10}} & +\frac{1}{2\sqrt{15}} & -\frac{2}{\sqrt{15}} & 0 & -\frac{1}{2}\sqrt{\frac{3}{5}} & +\frac{1}{\sqrt{6}} & 0 \\
+\frac{1}{2} & +\frac{1}{\sqrt{10}} & 0 & -\sqrt{\frac{3}{10}} & 0 & +\frac{1}{6\sqrt{5}} & 0 & 0 & +\frac{\sqrt{2}}{3} & -\frac{1}{3\sqrt{10}} & 0 & +\frac{1}{3} \\
-\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{5}} & 0 & +\frac{1}{2}\sqrt{\frac{3}{5}} & 0 & -\frac{1}{6\sqrt{10}} & 0 & 0 & +\frac{2}{3} & +\frac{1}{6\sqrt{5}} & 0 & +\frac{\sqrt{2}}{3} \\
0 & 0 & +\frac{1}{\sqrt{5}} & 0 & +\frac{1}{\sqrt{5}} & 0 & -\sqrt{\frac{2}{15}} & +\sqrt{\frac{2}{15}} & 0 & 0 & +\frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{2\sqrt{2}} & +\frac{1}{2\sqrt{5}} & 0 & -\frac{1}{2}\sqrt{\frac{3}{5}} & 0 & -\frac{3}{2\sqrt{10}} & 0 & 0 & 0 & +\frac{3}{2\sqrt{5}} & 0 & 0 \\
+\frac{1}{2} & -\frac{1}{2\sqrt{10}} & -\frac{1}{2}\sqrt{\frac{3}{5}} & +\frac{1}{2}\sqrt{\frac{3}{10}} & -\frac{1}{2}\sqrt{\frac{3}{5}} & -\frac{1}{2\sqrt{5}} & -\frac{1}{3}\sqrt{\frac{2}{5}} & +\frac{1}{3}\sqrt{\frac{2}{5}} & 0 & +\frac{1}{\sqrt{10}} & +\frac{1}{3} & 0 \\
-\frac{1}{2} & +\frac{1}{2\sqrt{10}} & -\frac{1}{2}\sqrt{\frac{3}{5}} & -\frac{1}{2}\sqrt{\frac{3}{10}} & -\frac{1}{2}\sqrt{\frac{3}{5}} & +\frac{1}{2\sqrt{5}} & -\frac{1}{3}\sqrt{\frac{2}{5}} & +\frac{1}{3}\sqrt{\frac{2}{5}} & 0 & -\frac{1}{\sqrt{10}} & +\frac{1}{3} & 0
\end{pmatrix}, \tag{A7}
$$

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to K^+\eta_1) \\ \mathcal{A}(B_d^0 \to K^0\eta_1) \\ \mathcal{A}(B_s^0 \to \pi^0\eta_1) \\ \mathcal{A}(B_s^0 \to \eta_8\eta_1) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{3}{2}} & +\frac{1}{2} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2} & -\frac{1}{2\sqrt{2}} & +\frac{1}{2} \\ 0 & +\frac{1}{\sqrt{2}} & 0 & +\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & 0 & +\frac{\sqrt{3}}{2} & 0 \end{pmatrix} \begin{pmatrix} \langle 8||\bar{3}_{I=0}||3\rangle \\ \langle 8||\bar{6}_{I=1}||3\rangle \\ \left\langle 8||\overline{15}_{I=0}||3\right\rangle \\ \left\langle 8||\overline{15}_{I=1}||3\right\rangle \end{pmatrix}, \tag{A8}$$

and

$$\mathcal{A}(B_s^0 \to \eta_1\eta_1) = (+1)\,\langle 1||\bar{3}_{I=0}||3\rangle\,. \tag{A9}$$

c. $\Delta S = -1$ ($\Delta I_3 = +1$, $\Delta Y = -\frac{4}{3}$)

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to \bar{K}^0\pi^+) \\ \mathcal{A}(B_d^0 \to \bar{K}^0\pi^0) \\ \mathcal{A}(B_d^0 \to \bar{K}^0\eta_8) \\ \mathcal{A}(B_d^0 \to K^-\pi^+) \\ \mathcal{A}(B_s^0 \to \bar{K}^0\bar{K}^0) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3}\sqrt{\frac{5}{2}} & -\frac{2}{3} & -\frac{1}{6} & -\frac{1}{2} \\ -\sqrt{\frac{3}{10}} & -\frac{2}{3\sqrt{5}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} & +\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{10}} & +\frac{1}{\sqrt{15}} & -\frac{1}{\sqrt{6}} & +\sqrt{\frac{2}{3}} & 0 \\ +\sqrt{\frac{3}{5}} & -\frac{1}{3\sqrt{10}} & -\frac{1}{3} & +\frac{1}{6} & +\frac{1}{2} \\ 0 & -\frac{\sqrt{5}}{3} & +\frac{\sqrt{2}}{3} & +\frac{\sqrt{2}}{3} & 0 \end{pmatrix} \begin{pmatrix} \left\langle 8||\overline{15}_{I=1}||3\right\rangle \\ \left\langle 27||\overline{15}_{I=1}||3\right\rangle \\ \langle 27||24_{I=1}||3\rangle \\ \left\langle 27||\overline{42}_{I=1}||3\right\rangle \\ \left\langle 27||\overline{42}_{I=2}||3\right\rangle \end{pmatrix}, \tag{A10}$$

and

$$\mathcal{A}(B_d^0 \to \bar{K}^0\eta_1) = (+1)\left\langle 1||\overline{15}_{I=1}||3\right\rangle. \tag{A11}$$

d. $\Delta S = +2$ ($\Delta I_3 = -\frac{1}{2}$, $\Delta Y = +\frac{5}{3}$)

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to K^+K^0) \\ \mathcal{A}(B_d^0 \to K^0K^0) \\ \mathcal{A}(B_s^0 \to K^0\pi^0) \\ \mathcal{A}(B_d^0 \to K^+\pi^-) \\ \mathcal{A}(B_s^0 \to K^0\eta_8) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3}\sqrt{\frac{5}{2}} & +\frac{2}{3} & -\frac{1}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} \\ 0 & -\frac{\sqrt{5}}{3} & -\frac{\sqrt{2}}{3} & -\frac{1}{3} & +\frac{1}{3} \\ -\sqrt{\frac{3}{10}} & +\frac{1}{6\sqrt{5}} & +\frac{\sqrt{2}}{3} & -\frac{1}{6} & +\frac{2}{3} \\ +\sqrt{\frac{3}{5}} & -\frac{1}{3\sqrt{10}} & +\frac{1}{3} & +\frac{1}{3\sqrt{2}} & +\frac{\sqrt{2}}{3} \\ -\frac{1}{\sqrt{10}} & -\frac{1}{2}\sqrt{\frac{3}{5}} & 0 & +\frac{\sqrt{3}}{2} & 0 \end{pmatrix} \begin{pmatrix} \left\langle 8||\overline{15}_{I=\frac{1}{2}}||3\right\rangle \\ \left\langle 27||\overline{15}_{I=\frac{1}{2}}||3\right\rangle \\ \left\langle 27||24_{I=\frac{3}{2}}||3\right\rangle \\ \left\langle 27||\overline{42}_{I=\frac{1}{2}}||3\right\rangle \\ \left\langle 27||\overline{42}_{I=\frac{3}{2}}||3\right\rangle \end{pmatrix}, \tag{A12}$$

and

$$\mathcal{A}(B_s^0 \to K^0\eta_1) = (+1)\left\langle 1||\overline{15}_{I=\frac{1}{2}}||3\right\rangle. \tag{A13}$$

e. $\Delta S = -2$ ($\Delta I_3 = +\frac{3}{2}$, $\Delta Y = -\frac{7}{3}$)

$$\mathcal{A}(B_d^0 \to \bar{K}^0 \bar{K}^0) = (+1) \left\langle 27 || \overline{42}_{I=\frac{3}{2}} || 3 \right\rangle . \tag{A14}$$

f. $\Delta S = +3$ ($\Delta I_3 = -1$, $\Delta Y = +\frac{8}{3}$)

$$\mathcal{A}(B_s^0 \to K^0 K^0) = (+1) \left\langle 27 || \overline{42}_{I=1} || 3 \right\rangle . \tag{A15}$$

2. $B \to D, P$

a. $\Delta S = 0$ ($\Delta I_3 = 0$, $\Delta Y = -\frac{2}{3}$)

For P an octet meson, denote $\mathbf{u}_{D,8}^{S=0} = \mathcal{O}_{D,8}^{S=0}\, \mathbf{v}_{D,8}^{S=0}$, where

$$\mathbf{u}_{D,8}^{S=0} = \begin{pmatrix} \mathcal{A}(B_u^+ \to D^+ \pi^0) \\ \mathcal{A}(B_u^+ \to D^+ \eta_8) \\ \mathcal{A}(B_u^+ \to D^0 \pi^+) \\ \mathcal{A}(B_u^+ \to D_s^+ \bar{K}^0) \\ \mathcal{A}(B_d^0 \to D^+ \pi^-) \\ \mathcal{A}(B_d^0 \to D^0 \pi^0) \\ \mathcal{A}(B_d^0 \to D^0 \eta_8) \\ \mathcal{A}(B_d^0 \to D_s^+ K^-) \\ \mathcal{A}(B_s^0 \to D^+ K^-) \\ \mathcal{A}(B_s^0 \to D^0 \bar{K}^0) \end{pmatrix}, \qquad \mathbf{v}_{D,8}^{S=0} = \begin{pmatrix} \langle \bar{3} || 3_{I=0} || 3 \rangle \\ \langle \bar{3} || \bar{6}_{I=1} || 3 \rangle \\ \langle 6 || 3_{I=0} || 3 \rangle \\ \langle 6 || 15_{I=0} || 3 \rangle \\ \langle 6 || 15_{I=1} || 3 \rangle \\ \left\langle \overline{15} || \bar{6}_{I=1} || 3 \right\rangle \\ \left\langle \overline{15} || 15_{I=0} || 3 \right\rangle \\ \left\langle \overline{15} || 15_{I=1} || 3 \right\rangle \\ \left\langle \overline{15} || \overline{24}_{I=1} || 3 \right\rangle \\ \left\langle \overline{15} || \overline{24}_{I=2} || 3 \right\rangle \end{pmatrix}, \tag{A16}$$

and $\mathcal{O}_{D,8}^{S=0} =$

$$\begin{pmatrix} -\frac{1}{4}\sqrt{\frac{3}{2}} & -\frac{1}{4}\sqrt{\frac{3}{2}} & -\frac{1}{4\sqrt{2}} & -\frac{1}{4\sqrt{2}} & -\frac{1}{4} & -\frac{1}{4}\sqrt{\frac{5}{2}} & +\frac{1}{4\sqrt{6}} & -\frac{\sqrt{3}}{4} & 0 & -\frac{1}{\sqrt{3}} \\ +\frac{1}{4\sqrt{2}} & +\frac{1}{4\sqrt{2}} & -\frac{1}{4}\sqrt{\frac{3}{2}} & -\frac{1}{4}\sqrt{\frac{3}{2}} & -\frac{\sqrt{3}}{4} & -\frac{1}{4}\sqrt{\frac{3}{10}} & -\frac{3}{4\sqrt{2}} & +\frac{1}{4} & -\frac{1}{\sqrt{5}} & 0 \\ +\frac{\sqrt{3}}{4} & +\frac{\sqrt{3}}{4} & +\frac{1}{4} & +\frac{1}{4} & +\frac{1}{2\sqrt{2}} & -\frac{3}{4\sqrt{5}} & -\frac{1}{4\sqrt{3}} & -\frac{1}{2\sqrt{6}} & -\frac{1}{\sqrt{30}} & -\frac{1}{\sqrt{6}} \\ -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & +\frac{1}{4} & +\frac{1}{4} & +\frac{1}{2\sqrt{2}} & -\frac{1}{4\sqrt{5}} & -\frac{\sqrt{3}}{4} & +\frac{1}{2\sqrt{6}} & -\sqrt{\frac{2}{15}} & 0 \\ -\frac{\sqrt{3}}{4} & +\frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{1}{4} & +\frac{1}{2\sqrt{2}} & -\frac{3}{4\sqrt{5}} & +\frac{1}{4\sqrt{3}} & -\frac{1}{2\sqrt{6}} & -\frac{1}{\sqrt{30}} & +\frac{1}{\sqrt{6}} \\ +\frac{1}{4}\sqrt{\frac{3}{2}} & -\frac{1}{4}\sqrt{\frac{3}{2}} & +\frac{1}{4\sqrt{2}} & +\frac{1}{4\sqrt{2}} & -\frac{1}{4} & -\frac{1}{4}\sqrt{\frac{5}{2}} & -\frac{1}{4\sqrt{6}} & -\frac{\sqrt{3}}{4} & 0 & +\frac{1}{\sqrt{3}} \\ +\frac{1}{4\sqrt{2}} & -\frac{1}{4\sqrt{2}} & -\frac{1}{4}\sqrt{\frac{3}{2}} & -\frac{1}{4}\sqrt{\frac{3}{2}} & +\frac{\sqrt{3}}{4} & +\frac{1}{4}\sqrt{\frac{3}{10}} & -\frac{3}{4\sqrt{2}} & -\frac{1}{4} & +\frac{1}{\sqrt{5}} & 0 \\ -\frac{\sqrt{3}}{4} & +\frac{\sqrt{3}}{4} & +\frac{1}{4} & +\frac{1}{4} & -\frac{1}{2\sqrt{2}} & +\frac{1}{4\sqrt{5}} & -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{6}} & +\sqrt{\frac{2}{15}} & 0 \\ 0 & 0 & -\frac{1}{2} & +\frac{1}{2} & 0 & -\frac{1}{\sqrt{5}} & 0 & +\frac{1}{\sqrt{6}} & +\sqrt{\frac{2}{15}} & 0 \\ 0 & 0 & -\frac{1}{2} & +\frac{1}{2} & 0 & +\frac{1}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{15}} & 0 \end{pmatrix}, \tag{A17}$$

and

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to D^+\eta_1) \\ \mathcal{A}(B_d^0 \to D^0\eta_1) \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \langle\bar{3}||3_{I=0}||3\rangle \\ \langle\bar{3}||\bar{6}_{I=1}||3\rangle \end{pmatrix}. \tag{A18}$$

b. $\Delta S = +1$ $(\Delta I_3 = -\frac{1}{2},\, \Delta Y = +\frac{1}{3})$

For P an octet meson, denote $\mathbf{u}_{D,8}^{S=1} = \mathcal{O}_{D,8}^{S=1}\,\mathbf{v}_{D,8}^{S=1}$, where

$$\mathbf{u}_{D,8}^{S=1} = \begin{pmatrix} \mathcal{A}(B_u^+ \to D^+K^0) \\ \mathcal{A}(B_u^+ \to D^0K^+) \\ \mathcal{A}(B_u^+ \to D_s^+\pi^0) \\ \mathcal{A}(B_u^+ \to D_s^+\eta_8) \\ \mathcal{A}(B_d^0 \to D^0K^0) \\ \mathcal{A}(B_d^0 \to D_s^+\pi^-) \\ \mathcal{A}(B_s^0 \to D^+\pi^-) \\ \mathcal{A}(B_s^0 \to D^0\pi^0) \\ \mathcal{A}(B_s^0 \to D^0\eta_8) \\ \mathcal{A}(B_s^0 \to D_s^+K^-) \end{pmatrix}, \qquad \mathbf{v}_{D,8}^{S=1} = \begin{pmatrix} \left\langle\bar{3}||3_{I=\frac{1}{2}}||3\right\rangle \\ \left\langle\bar{3}||\bar{6}_{I=\frac{1}{2}}||3\right\rangle \\ \left\langle 6||3_{I=\frac{1}{2}}||3\right\rangle \\ \left\langle 6||15_{I=\frac{1}{2}}||3\right\rangle \\ \left\langle 6||15_{I=\frac{3}{2}}||3\right\rangle \\ \left\langle\overline{15}||\bar{6}_{I=\frac{1}{2}}||3\right\rangle \\ \left\langle\overline{15}||15_{I=\frac{1}{2}}||3\right\rangle \\ \left\langle\overline{15}||15_{I=\frac{3}{2}}||3\right\rangle \\ \left\langle\overline{15}||\overline{24}_{I=\frac{1}{2}}||3\right\rangle \\ \left\langle\overline{15}||\overline{24}_{I=\frac{3}{2}}||3\right\rangle \end{pmatrix}, \tag{A19}$$

and $\mathcal{O}_{D,8}^{S=1} =$

$$\begin{pmatrix}
+\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{1}{4\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{4\sqrt{5}} & -\frac{5}{12} & +\frac{1}{3\sqrt{2}} & +\frac{1}{3\sqrt{5}} & -\frac{1}{3} \\
-\frac{\sqrt{3}}{4} & +\frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{1}{4\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{3}{4\sqrt{5}} & +\frac{1}{12} & +\frac{1}{3\sqrt{2}} & -\frac{2}{3\sqrt{5}} & -\frac{1}{3} \\
0 & 0 & +\frac{1}{2\sqrt{2}} & +\frac{1}{2\sqrt{6}} & +\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{10}} & -\frac{1}{3\sqrt{2}} & +\frac{1}{3} & -\frac{1}{3\sqrt{10}} & -\frac{\sqrt{2}}{3} \\
+\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{3}{10}} & +\frac{1}{2}\sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{3}{10}} & 0 \\
0 & 0 & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} & +\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & -\frac{1}{3} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{5}} & +\frac{1}{3} \\
0 & 0 & +\frac{1}{2} & +\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & -\frac{1}{3} & -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{5}} & +\frac{1}{3} \\
+\frac{\sqrt{3}}{4} & +\frac{\sqrt{3}}{4} & -\frac{1}{4} & +\frac{\sqrt{3}}{4} & 0 & +\frac{1}{4\sqrt{5}} & -\frac{1}{12} & +\frac{\sqrt{2}}{3} & -\frac{1}{3\sqrt{5}} & +\frac{1}{3} \\
-\frac{1}{4}\sqrt{\frac{3}{2}} & -\frac{1}{4}\sqrt{\frac{3}{2}} & +\frac{1}{4\sqrt{2}} & -\frac{1}{4}\sqrt{\frac{3}{2}} & 0 & -\frac{1}{4\sqrt{10}} & +\frac{1}{12\sqrt{2}} & +\frac{2}{3} & +\frac{1}{3\sqrt{10}} & +\frac{\sqrt{2}}{3} \\
-\frac{1}{4\sqrt{2}} & -\frac{1}{4\sqrt{2}} & -\frac{1}{4}\sqrt{\frac{3}{2}} & +\frac{3}{4\sqrt{2}} & 0 & -\frac{3}{4}\sqrt{\frac{3}{10}} & +\frac{1}{4}\sqrt{\frac{3}{2}} & 0 & +\sqrt{\frac{3}{10}} & 0 \\
+\frac{\sqrt{3}}{4} & +\frac{\sqrt{3}}{4} & +\frac{1}{4} & -\frac{\sqrt{3}}{4} & 0 & -\frac{3}{4\sqrt{5}} & +\frac{1}{4} & 0 & +\frac{1}{\sqrt{5}} & 0
\end{pmatrix}, \tag{A20}$$

and

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to D_s^+ \eta_1) \\ \mathcal{A}(B_s^0 \to D^0 \eta_1) \end{pmatrix} = \begin{pmatrix} +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \left\langle \bar{3} || 3_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle \bar{3} || \bar{6}_{I=\frac{1}{2}} || 3 \right\rangle \end{pmatrix}. \tag{A21}$$

c. $\Delta S = -1$ $(\Delta I_3 = +\frac{1}{2},\, \Delta Y = -\frac{5}{3})$

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to D^+ \bar{K}^0) \\ \mathcal{A}(B_d^0 \to D^+ K^-) \\ \mathcal{A}(B_d^0 \to D^0 \bar{K}^0) \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{3}} \\ +\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \left\langle 6 || 15_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle \overline{15} || 15_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle \overline{15} || \overline{24}_{I=\frac{3}{2}} || 3 \right\rangle \end{pmatrix}. \tag{A22}$$

d. $\Delta S = +2$ $(\Delta I_3 = -1,\, \Delta Y = +\frac{4}{3})$

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to D_s^+ K^0) \\ \mathcal{A}(B_s^0 \to D^0 K^0) \\ \mathcal{A}(B_s^0 \to D_s^+ \pi^-) \end{pmatrix} = \begin{pmatrix} 0 & +\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ +\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \langle 6 || 15_{I=1} || 3 \rangle \\ \left\langle \overline{15} || 15_{I=1} || 3 \right\rangle \\ \left\langle \overline{15} || \overline{24}_{I=1} || 3 \right\rangle \end{pmatrix}. \tag{A23}$$

3. $B \to \bar{D}, P$

a. $\Delta S = 0$ ($\Delta I_3 = +1$, $\Delta Y = 0$)

For P an octet meson, denote $\mathbf{u}^{S=0}_{\bar{D},8} = \mathcal{O}^{S=0}_{\bar{D},8}\,\mathbf{v}^{S=0}_{\bar{D},8}$, where

$$\mathbf{u}^{S=0}_{\bar{D},8} = \begin{pmatrix} \mathcal{A}(B_u^+ \to \bar{D}^0\pi^+) \\ \mathcal{A}(B_d^0 \to D^-\pi^+) \\ \mathcal{A}(B_d^0 \to \bar{D}^0\pi^0) \\ \mathcal{A}(B_d^0 \to \bar{D}^0\eta_8) \\ \mathcal{A}(B_d^0 \to D_s^- K^+) \\ \mathcal{A}(B_s^0 \to \bar{D}^0\bar{K}^0) \\ \mathcal{A}(B_s^0 \to D_s^-\pi^+) \end{pmatrix}, \qquad \mathbf{v}^{S=0}_{\bar{D},8} = \begin{pmatrix} \langle 3||8_{I=1}||3\rangle \\ \langle \bar{6}||8_{I=1}||3\rangle \\ \left\langle \bar{6}||\overline{10}_{I=1}||3\right\rangle \\ \langle 15||8_{I=1}||3\rangle \\ \langle 15||10_{I=1}||3\rangle \\ \langle 15||27_{I=1}||3\rangle \\ \langle 15||27_{I=2}||3\rangle \end{pmatrix}, \tag{A24}$$

and $\mathcal{O}^{S=0}_{\bar{D},8} =$

$$\begin{pmatrix} 0 & 0 & 0 & -2\sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{5}} & -\frac{1}{2} \\ -\frac{1}{2}\sqrt{\frac{3}{2}} & +\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2}\sqrt{\frac{3}{10}} & 0 & -\frac{1}{2\sqrt{5}} & +\frac{1}{2} \\ +\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{6}} & +\frac{1}{2\sqrt{3}} & -\frac{1}{4}\sqrt{\frac{5}{3}} & -\frac{1}{2\sqrt{3}} & 0 & +\frac{1}{\sqrt{2}} \\ +\frac{1}{4} & +\frac{1}{2\sqrt{2}} & -\frac{1}{2} & +\frac{1}{4\sqrt{5}} & -\frac{1}{2} & +\sqrt{\frac{3}{10}} & 0 \\ -\frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2\sqrt{3}} & +\frac{1}{\sqrt{6}} & +\frac{1}{2\sqrt{30}} & -\frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{5}} & 0 \\ 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{15}} & +\frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{5}} & 0 \\ 0 & +\frac{1}{\sqrt{3}} & +\frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{15}} & +\frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{5}} & 0 \end{pmatrix}, \tag{A25}$$

and

$$\mathcal{A}(B_d^0 \to \bar{D}^0\eta_1) = (+1)\,\langle 3||8_{I=1}||3\rangle\,. \tag{A26}$$

b. $\Delta S = +1$ ($\Delta I_3 = +\frac{1}{2}$, $\Delta Y = +1$)

For P an octet meson, denote $\mathbf{u}^{S=1}_{\bar{D},8} = \mathcal{O}^{S=1}_{\bar{D},8}\,\mathbf{v}^{S=1}_{\bar{D},8}$, where

$$
\mathbf{u}^{S=1}_{\bar{D},8} = \begin{pmatrix} \mathcal{A}(B_u^+ \to \bar{D}^0 K^+) \\ \mathcal{A}(B_d^0 \to D^- K^+) \\ \mathcal{A}(B_d^0 \to \bar{D}^0 K^0) \\ \mathcal{A}(B_s^0 \to D^- \pi^+) \\ \mathcal{A}(B_s^0 \to \bar{D}^0 \pi^0) \\ \mathcal{A}(B_s^0 \to \bar{D}^0 \eta_8) \\ \mathcal{A}(B_s^0 \to D_s^- K^+) \end{pmatrix}, \qquad \mathbf{v}^{S=1}_{\bar{D},8} = \begin{pmatrix} \left\langle 3 || 8_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle \bar{6} || 8_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle \bar{6} || \overline{10}_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle 15 || 8_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle 15 || 10_{I=\frac{3}{2}} || 3 \right\rangle \\ \left\langle 15 || 27_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle 15 || 27_{I=\frac{3}{2}} || 3 \right\rangle \end{pmatrix}, \tag{A27}
$$

and $\mathcal{O}^{S=1}_{\bar{D},8} =$

$$
\begin{pmatrix}
0 & 0 & 0 & -2\sqrt{\frac{2}{15}} & +\frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{6}} \\
0 & +\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{30}} & +\frac{1}{\sqrt{6}} \\
0 & -\frac{1}{\sqrt{3}} & +\frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{15}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{30}} & +\frac{1}{\sqrt{6}} \\
-\frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{\sqrt{6}} & +\frac{1}{2\sqrt{30}} & +\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{30}} & +\frac{1}{\sqrt{6}} \\
+\frac{\sqrt{3}}{4} & +\frac{1}{2\sqrt{6}} & +\frac{1}{2\sqrt{3}} & -\frac{1}{4\sqrt{15}} & +\frac{1}{\sqrt{3}} & +\frac{1}{2\sqrt{15}} & +\frac{1}{\sqrt{3}} \\
+\frac{1}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} & -\frac{3}{4\sqrt{5}} & 0 & +\frac{3}{2\sqrt{5}} & 0 \\
-\frac{1}{2}\sqrt{\frac{3}{2}} & +\frac{1}{2\sqrt{3}} & +\frac{1}{\sqrt{6}} & -\frac{1}{2}\sqrt{\frac{3}{10}} & 0 & +\sqrt{\frac{3}{10}} & 0
\end{pmatrix}, \tag{A28}
$$

and

$$
\mathcal{A}(B_s^0 \to \bar{D}^0 \eta_1) = (+1) \left\langle 3 || 8_{I=\frac{1}{2}} || 3 \right\rangle . \tag{A29}
$$

c. $\Delta S = -1$ ($\Delta I_3 = +\frac{3}{2}$, $\Delta Y = -1$)

$$
\begin{pmatrix} \mathcal{A}(B_d^0 \to \bar{D}^0 \bar{K}^0) \\ \mathcal{A}(B_d^0 \to D_s^- \pi^+) \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \left\langle \bar{6} || \overline{10}_{I=\frac{3}{2}} || 3 \right\rangle \\ \left\langle 15 || 27_{I=\frac{3}{2}} || 3 \right\rangle \end{pmatrix} . \tag{A30}
$$

d. $\Delta S = +2$ ($\Delta I_3 = 0$, $\Delta Y = +2$)

$$
\begin{pmatrix} \mathcal{A}(B_s^0 \to D^- K^+) \\ \mathcal{A}(B_s^0 \to \bar{D}^0 K^0) \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \left\langle \bar{6} || \overline{10}_{I=0} || 3 \right\rangle \\ \langle 15 || 27_{I=1} || 3 \rangle \end{pmatrix} . \tag{A31}
$$

4. $B \to D, \bar{D}$

a. $\Delta S = 0$ $(\Delta I_3 = +\frac{1}{2},\, \Delta Y = -\frac{1}{3})$

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to D^+ \bar{D}^0) \\ \mathcal{A}(B_d^0 \to D^+ D^-) \\ \mathcal{A}(B_d^0 \to D^0 \bar{D}^0) \\ \mathcal{A}(B_d^0 \to D_s^+ D_s^-) \\ \mathcal{A}(B_s^0 \to D^+ D_s^-) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2} & -\frac{1}{2\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ +\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{6}} & +\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{6}} & -\frac{1}{2} & +\frac{1}{2\sqrt{6}} & +\frac{1}{\sqrt{3}} \\ +\frac{1}{\sqrt{3}} & +\frac{1}{2\sqrt{6}} & -\frac{1}{2} & +\frac{1}{2}\sqrt{\frac{3}{2}} & 0 \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}} & +\frac{1}{2} & +\frac{1}{2}\sqrt{\frac{3}{2}} & 0 \end{pmatrix} \begin{pmatrix} \left\langle 1 || \bar{3}_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle 8 || \bar{3}_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle 8 || 6_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle 8 || \overline{15}_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle 8 || \overline{15}_{I=\frac{3}{2}} || 3 \right\rangle \end{pmatrix} . \tag{A32}$$

b. $\Delta S = +1$ $(\Delta I_3 = 0,\, \Delta Y = +\frac{2}{3})$

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to D_s^+ \bar{D}^0) \\ \mathcal{A}(B_d^0 \to D_s^+ D^-) \\ \mathcal{A}(B_s^0 \to D^+ D^-) \\ \mathcal{A}(B_s^0 \to D^0 \bar{D}^0) \\ \mathcal{A}(B_s^0 \to D_s^+ D_s^-) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{\frac{3}{2}} & +\frac{1}{2} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} \\ 0 & -\frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2} & -\frac{1}{2\sqrt{2}} & +\frac{1}{2} \\ +\frac{1}{\sqrt{3}} & +\frac{1}{2\sqrt{6}} & +\frac{1}{2} & -\frac{1}{2\sqrt{2}} & +\frac{1}{2} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{2\sqrt{6}} & +\frac{1}{2} & +\frac{1}{2\sqrt{2}} & +\frac{1}{2} \\ +\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & 0 & +\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \langle 1 || \bar{3}_{I=0} || 3 \rangle \\ \langle 8 || \bar{3}_{I=0} || 3 \rangle \\ \langle 8 || 6_{I=1} || 3 \rangle \\ \left\langle 8 || \overline{15}_{I=0} || 3 \right\rangle \\ \left\langle 8 || \overline{15}_{I=1} || 3 \right\rangle \end{pmatrix} . \tag{A33}$$

c. $\Delta S = -1$ $(\Delta I_3 = +1,\, \Delta Y = -\frac{4}{3})$

$$\mathcal{A}(B_d^0 \to D^+ D_s^-) = (+1) \left\langle 8 || \overline{15}_{I=1} || 3 \right\rangle . \tag{A34}$$

d. $\Delta S = +2$ $(\Delta I_3 = -\frac{1}{2},\, \Delta Y = +\frac{5}{3})$

$$\mathcal{A}(B_s^0 \to D_s^+ D^-) = (+1) \left\langle 8 || \overline{15}_{I=\frac{1}{2}} || 3 \right\rangle . \tag{A35}$$

5. $B \to \eta_c, P$

a. $\Delta S = 0$ ($\Delta I_3 = +\frac{1}{2}$, $\Delta Y = -\frac{1}{3}$)

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to \eta_c \pi^+) \\ \mathcal{A}(B_d^0 \to \eta_c \pi^0) \\ \mathcal{A}(B_d^0 \to \eta_c \eta_8) \\ \mathcal{A}(B_s^0 \to \eta_c \bar{K}^0) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2} & -\frac{1}{2\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{2\sqrt{2}} & -\frac{1}{4\sqrt{3}} & +\sqrt{\frac{2}{3}} \\ +\frac{1}{4} & -\frac{1}{2}\sqrt{\frac{3}{2}} & +\frac{3}{4} & 0 \\ -\frac{1}{2}\sqrt{\frac{3}{2}} & +\frac{1}{2} & +\frac{1}{2}\sqrt{\frac{3}{2}} & 0 \end{pmatrix} \begin{pmatrix} \left\langle 8 || \bar{3}_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle 8 || 6_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle 8 || \overline{15}_{I=\frac{1}{2}} || 3 \right\rangle \\ \left\langle 8 || \overline{15}_{I=\frac{3}{2}} || 3 \right\rangle \end{pmatrix}, \tag{A36}$$

and

$$\mathcal{A}(B_d^0 \to \eta_c \eta_1) = (+1) \left\langle 1 || \bar{3}_{I=\frac{1}{2}} || 3 \right\rangle . \tag{A37}$$

b. $\Delta S = +1$ ($\Delta I_3 = 0$, $\Delta Y = +\frac{2}{3}$)

$$\begin{pmatrix} \mathcal{A}(B_u^+ \to \eta_c K^+) \\ \mathcal{A}(B_d^0 \to \eta_c K^0) \\ \mathcal{A}(B_s^0 \to \eta_c \pi^0) \\ \mathcal{A}(B_s^0 \to \eta_c \eta_8) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{\frac{3}{2}} & +\frac{1}{2} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2}\sqrt{\frac{3}{2}} & -\frac{1}{2} & -\frac{1}{2\sqrt{2}} & +\frac{1}{2} \\ 0 & +\frac{1}{\sqrt{2}} & 0 & +\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & 0 & +\frac{\sqrt{3}}{2} & 0 \end{pmatrix} \begin{pmatrix} \langle 8 || \bar{3}_{I=0} || 3 \rangle \\ \langle 8 || 6_{I=1} || 3 \rangle \\ \left\langle 8 || \overline{15}_{I=0} || 3 \right\rangle \\ \left\langle 8 || \overline{15}_{I=1} || 3 \right\rangle \end{pmatrix}, \tag{A38}$$

and

$$\mathcal{A}(B_s^0 \to \eta_c \eta_1) = (+1) \langle 1 || \bar{3}_{I=0} || 3 \rangle . \tag{A39}$$

c. $\Delta S = -1$ ($\Delta I_3 = +1$, $\Delta Y = -\frac{4}{3}$)

$$\mathcal{A}(B_d^0 \to \eta_c \bar{K}^0) = (+1) \left\langle 8 || \overline{15}_{I=1} || 3 \right\rangle . \tag{A40}$$

d. $\Delta S = +2$ ($\Delta I_3 = -\frac{1}{2}$, $\Delta Y = +\frac{5}{3}$)

$$\mathcal{A}(B_s^0 \to \eta_c K^0) = (+1) \left\langle 8 || \overline{15}_{I=\frac{1}{2}} || 3 \right\rangle . \tag{A41}$$